

Muon scattering: Theory Experiment Monte Carlo

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MuonsInc workshop

Theory

Rutherford Scattering:

$$Nt\sigma[\chi] \chi d\chi = 2\chi_{c1}^2 \chi q[\chi] / \chi^4 d\chi$$

$$\chi_{c1}^2 = 4\pi Nt e^4 Z^2 / (pc\beta)^2$$

- Nt The number of atoms/cm² in the scattering target,
- Z The charge of the nucleus,
- β The velocity of the beam particle,
- $q(\chi)$ The form factor of the scattering center,
- χ_{c1} A parameter that defines the target and beam. Note that we have used Z^2 in the definition instead of $Z(Z+1)$ which is used in B10 and we distinguish this by an additional 1 in the subscript.
- p the momentum of the beam particle.

In the following, B*** refers to formulas in Bethe's article , PR 1256, 89 , 1953

χ_{c1} is the angle where $\text{Prob}[\chi \geq \chi_{c1}] = 1$ and is a property of the target material and thickness.

Trouble calculating the rms scattering angle for Rutherford scattering

$$\chi_{rms} = \int_0^{\infty} \chi^2 \frac{\chi_{c1}^2}{\chi^4} q(\chi) \chi d\chi \Rightarrow \infty$$

The form factor solves the small χ

divergence but there is a log divergence as $\chi \Rightarrow \infty$.

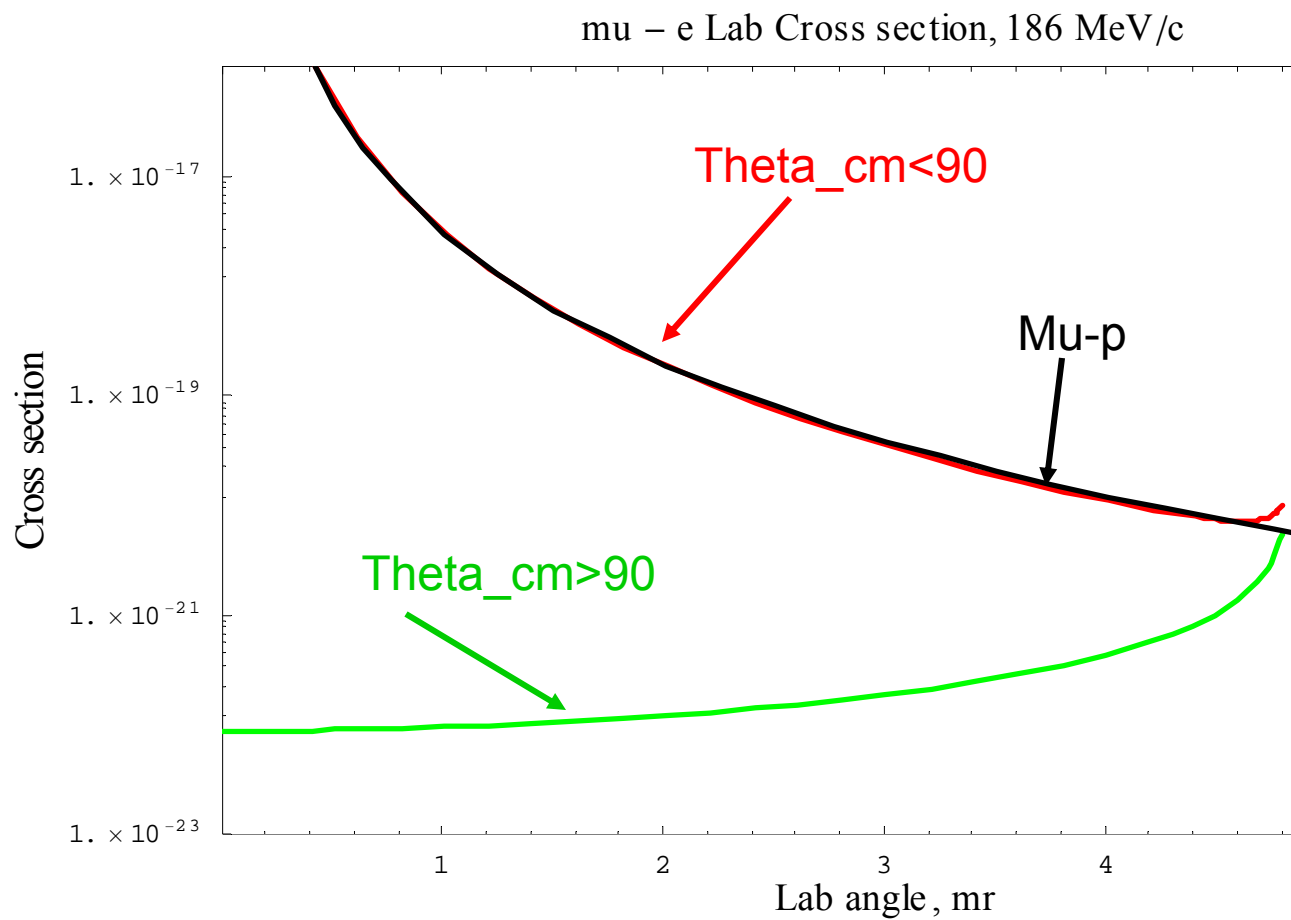
The solution is formulated thru the use of the transport equation>

$$\partial_t \mathbf{f}[\theta, t] = -N \mathbf{f}[\theta, t] \int \sigma[\chi] \chi d\chi + N \int \mathbf{f}[\theta', t] \sigma[\chi] d\chi$$

Bethe makes a Fourier Bessel transform and solves the equation:

$$\mathbf{f}[\theta, t] = \int_0^{\infty} \eta J_0[\theta \eta] \mathbf{Exp}[-Nt \int_0^{\infty} \sigma[\chi] \chi (1 - J_0[\eta \chi]) d\chi] d\eta$$

The PDG formula for multiple scattering gives the gaussian center but eventually falls below the single scattering cross section. The above formulation connects the small angle gaussian with the large angle single scattering tail.



Black is mu-p scattering. The red and green are the two branches of the mu-e scattering

THE CROSS SECTION :

It has been customary to:

1. Use the Thomas Fermi model for the atom. OK large Z. Bad for small Z.
2. Old guys knew that scattering off the electrons was nearly the same as for a proton and so they used $Z(Z+1)$ in the Rutherford scattering law.

CORRECTIONS:

1. Use the correct wave functions for hydrogen and use the elastic and inelastic form factors that are available in x-ray scattering tables to give the correct cross section formula:

$$\sigma[\chi] = 2 \chi_{c1}^2 \left[\frac{(\sigma_{el}[\chi] + \frac{1}{Z} \sigma_{inel}[\chi])}{(1 + \frac{1}{Z})} \right] \left[1 + \frac{1}{Z} \right] / \chi^4$$

Hydrogen:

$\sigma_{el}[t] = (1 - f[t])^2 \quad ; \quad \sigma_{inel}[t] = 1 - f[t]^2$
$f[t] = \frac{1}{(1 - a_0^2 t / 4)^2}$
Hydrogen Bohr radius a_0

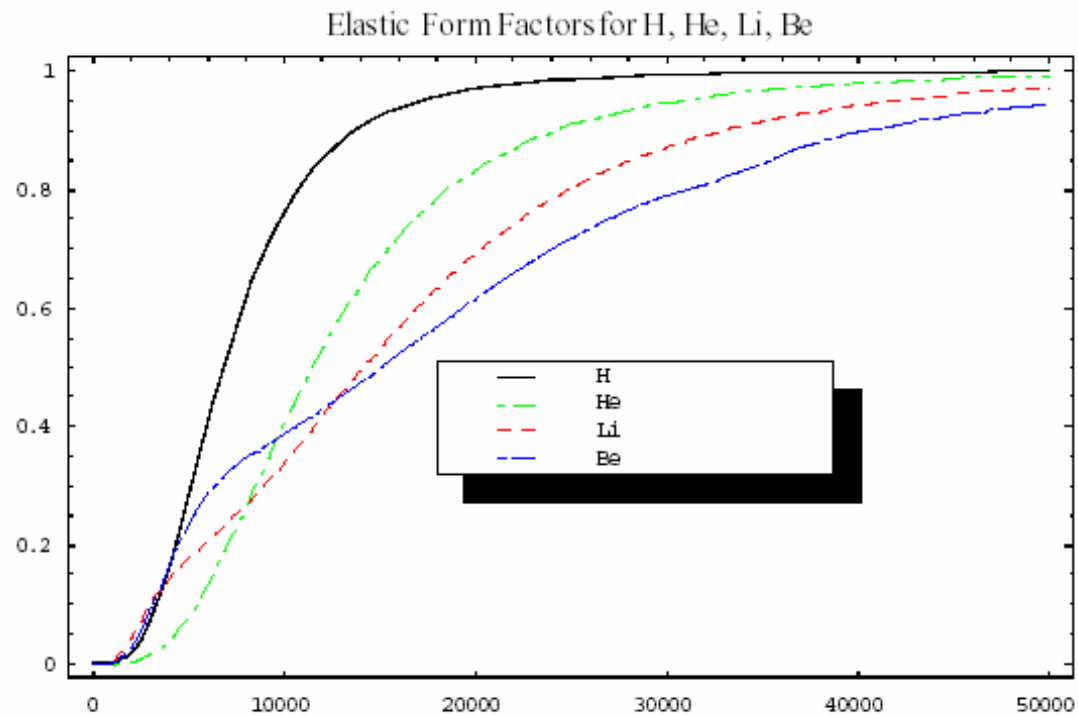


Fig.2. The elastic form factors for H, He, Li, and Be as a function of momentum transfer in units of eV/c .

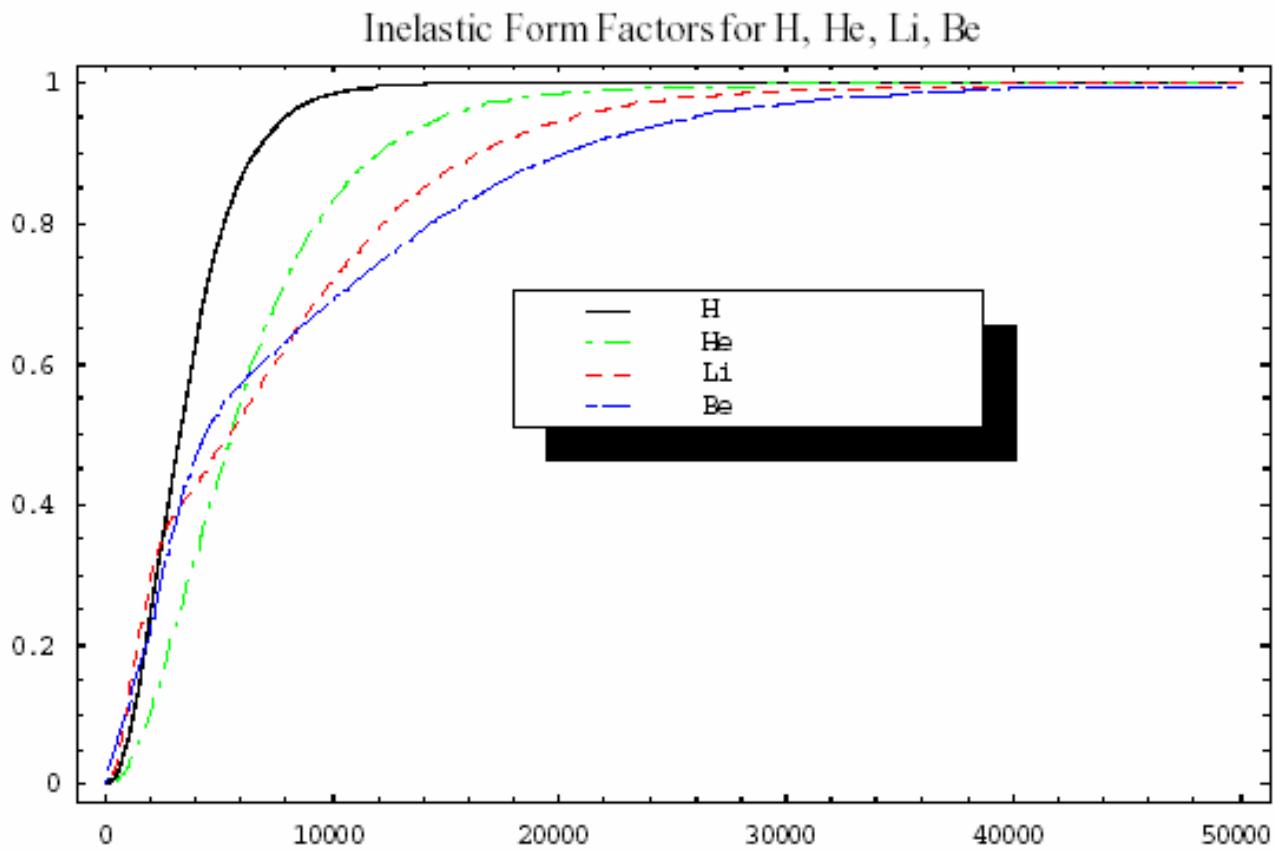


Fig.3. Inelastic form factors for H, He, Li, and Be as a function of momentum transfer in units of eV/c.

$$f[\theta, t] = \int_0^\infty \eta J_0[\theta \eta] \text{Exp}\left[-Nt \int_0^\infty \sigma[\chi] \chi (1 - J_0[\eta \chi]) d\chi\right] d\eta$$

OUTLINE OF PROCEDURE:

1. Carry out integrals over cross sections numerically for small t. For large t, the form factors =1 and the integrals can be done analytically. The integrals are independent of thickness and depend only on the element and thus yield a screening angle $\chi_a(z)$. Hence a table of values can be made. Using this parameter, Bethe and Moliere put the above integral in the form:

$$f[\theta] \theta d\theta = \frac{\theta d\theta}{B\chi_c^2} \int_0^\infty u J_0\left[\frac{u\theta}{\sqrt{B}\chi_c^2}\right] e^{\left(-\frac{1}{4}u^2 + \frac{1}{4B}u^2 \ln\left[\frac{1}{4}u^2\right]\right)} du$$

$$b = \ln\left[\frac{\chi_c^2}{\chi_a^2}\right] + 1 - 2 * .577 \qquad B - \ln[B] = b$$

The first term in the exponent leads to a gaussian and the second term approaches the χ^{-4} law for large angles.

2. There is an additional modification for the inelastic part.
 1. Unlike the elastic part, the inelastic integral diverges at 0. However, it is also cut off at the first excitation level of the atom. The value is very insensitive to this cutoff.
 2. The inelastic integral does not go to large angles. It must be cutoff at :

$$\frac{m_e}{m_\mu} = 4.8 \quad \text{mr}$$

Provided the multiple scattering is greater than 4.8 mr, the electrons will contribute only to the gaussian core! This is the case we find most often in muon cooling systems.

$$\text{bel} = \int_0^{\theta_0} \frac{q^{\text{inel}}[\chi]}{\chi} d\chi$$

3. The elastic is treated as before and a combined b generated

$$b' = \ln\left[\frac{\chi_{c1}}{\chi_a^{\text{el}}}\right]^2 + 1 - 2 * .577 + \frac{2 * \text{bel}}{z} \qquad B' - \log[B'] = b'$$

$$f[\theta] \theta d\theta = \frac{\theta d\theta}{B\chi_c^2} \int_0^\infty u J_0 \left[\frac{u\theta}{\sqrt{B\chi_c^2}} \right] e^{\left(-\frac{1}{4} u^2 + \frac{1}{4B} u^2 \ln\left[\frac{1}{4} u^2\right]\right)} du$$

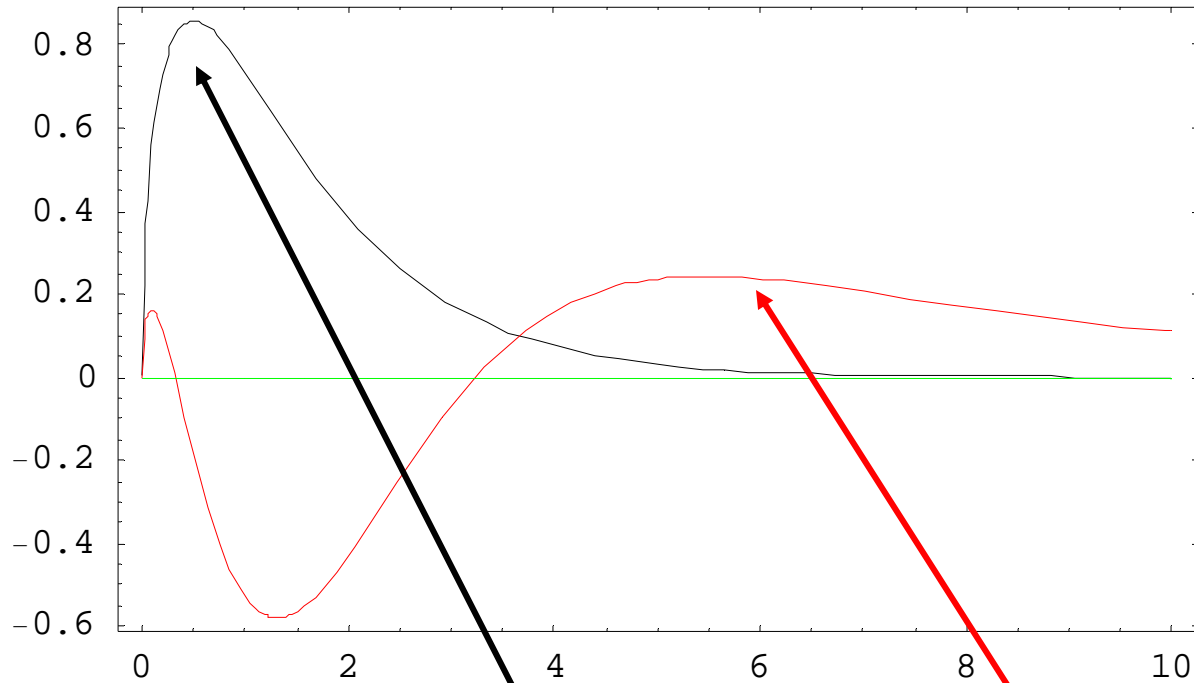
Where B' is used in place of B. Bethe expands the above into a series, but now days it is easy to numerically evaluate it directly.

$$f[p_t] p_t dp_t = \frac{p_t dp_t}{B p_c^2} \left(\underbrace{f^{(0)} \left[\frac{p_t^2}{B p_c^2} \right]}_{\text{Gaussian}} + \frac{1}{B} \underbrace{f^{(1)} \left[\frac{p_t^2}{B p_c^2} \right]}_{\text{Transition to single scattering}} + \frac{1}{B^2} \dots \dots \right)$$

$$f^{(0)}[x] = 2 e^{-x}$$

$$f^{(1)}[x] = 2 e^{-x} (x - 1) (\text{LogIntegral}[e^x] - \ln[x]) - 2 (1 - 2 e^{-x})$$

Black $y^{0.5}$ $g[y]$ Gaussian component
 Red $y^{0.5}$ $g1[y]$



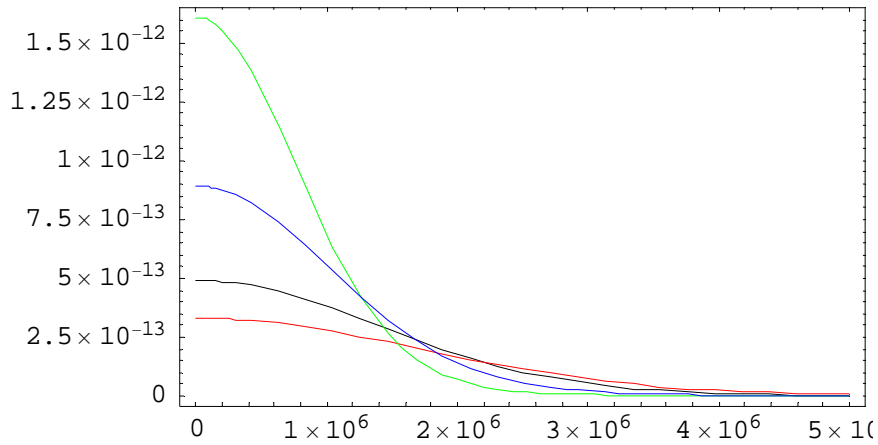
$$f[p_t] p_t dp_t = \frac{p_t dp_t}{B p_c^2} \left(f^{(0)} \left[\frac{p_t^2}{B p_c^2} \right] + \frac{1}{B} f^{(1)} \left[\frac{p_t^2}{B p_c^2} \right] + \frac{1}{B^2} \dots \dots \right)$$

gaussian

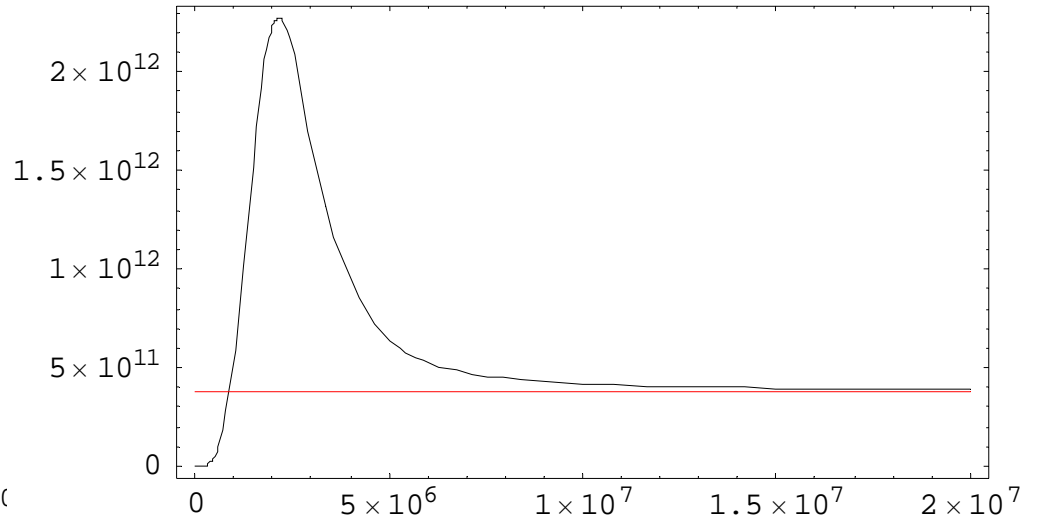
Approaches Rutherford (Z^2)

Some predictions for targets of the MUSCAT experiment at P=172 MeV/c

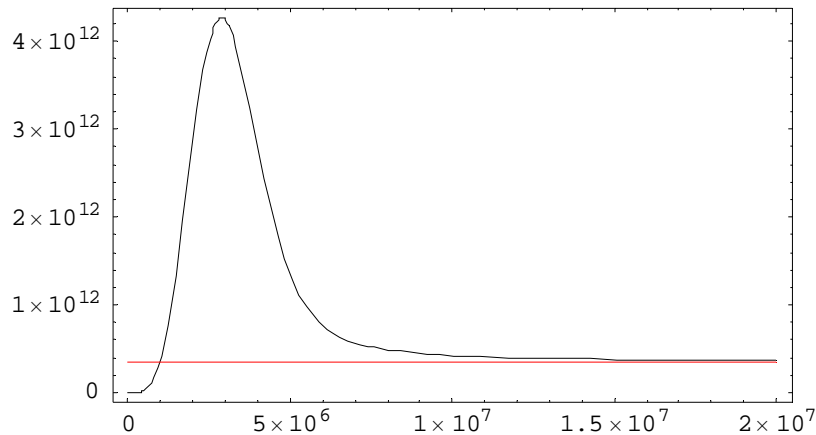
Black H .82 grms/cm2
 Red H 1.2 grms/cm2
 Green Li .34 grms/cm^2
 Blue C ..42 grms/cm2



$pt^4 * \sigma[pt]$ compared to Rutherford scattering
 The scattering is dominated by single scattering for $pt > 10$ MeV/c. Horz scale pt in eV/c. Carbon



$pt^4 * \sigma[pt]$ compared to Rutherford scattering
 The scattering is dominated by single scattering for $pt > 10$ MeV/c. Horz scale pt in eV/c. Thin H



The red line is a constant given by:

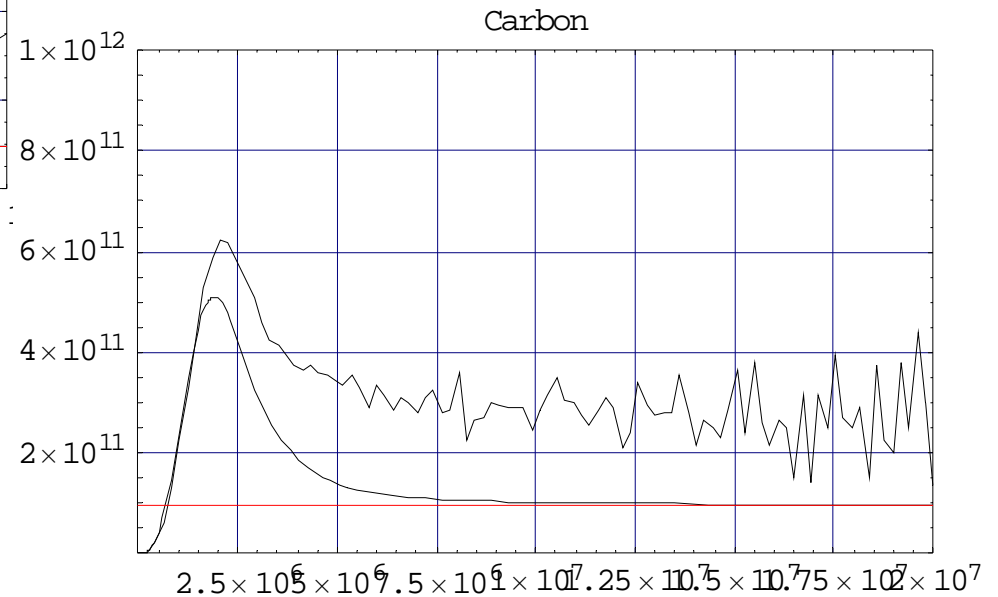
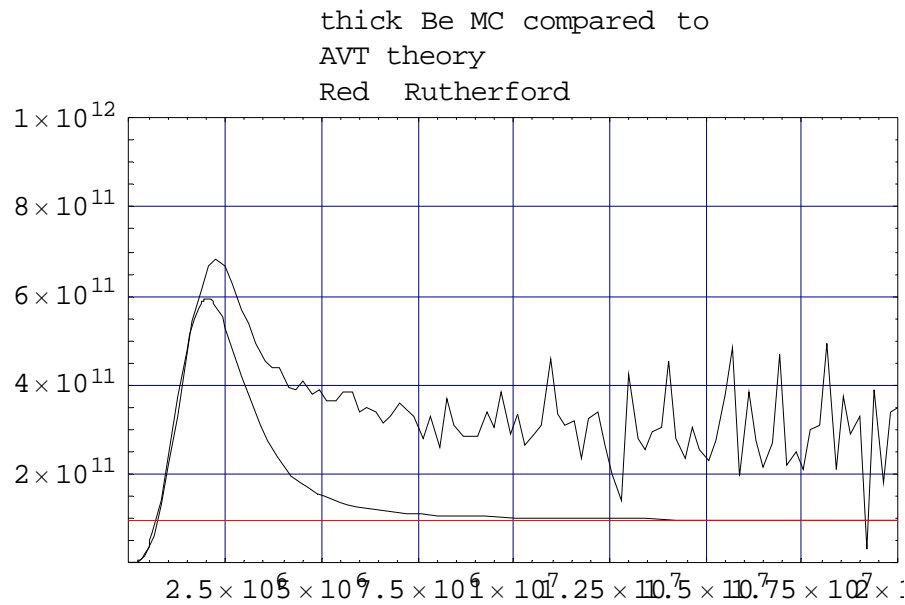
$$p_t^4 \sigma_{Rutherford} = p_t^4 \left(2 \frac{p_{c1}^2}{p_t^4} \right) = 2 p_{c1}^2$$

$$p_{c1}^2 = \frac{4 \pi N t e^4 z^2}{[c\beta]^2}$$

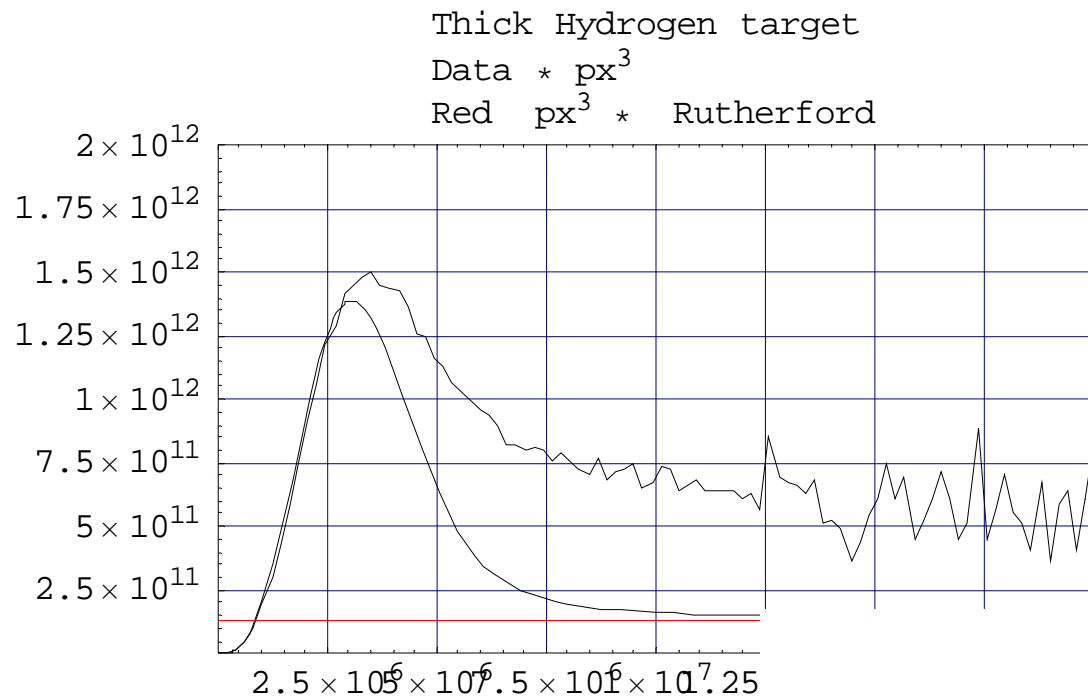
Note single scattering predominates after : $p_t > 5 - 7 \text{ Mev} / C$

The large angle scattering serves as a test for any theory, Monte Carlo calculation, or Experiment!

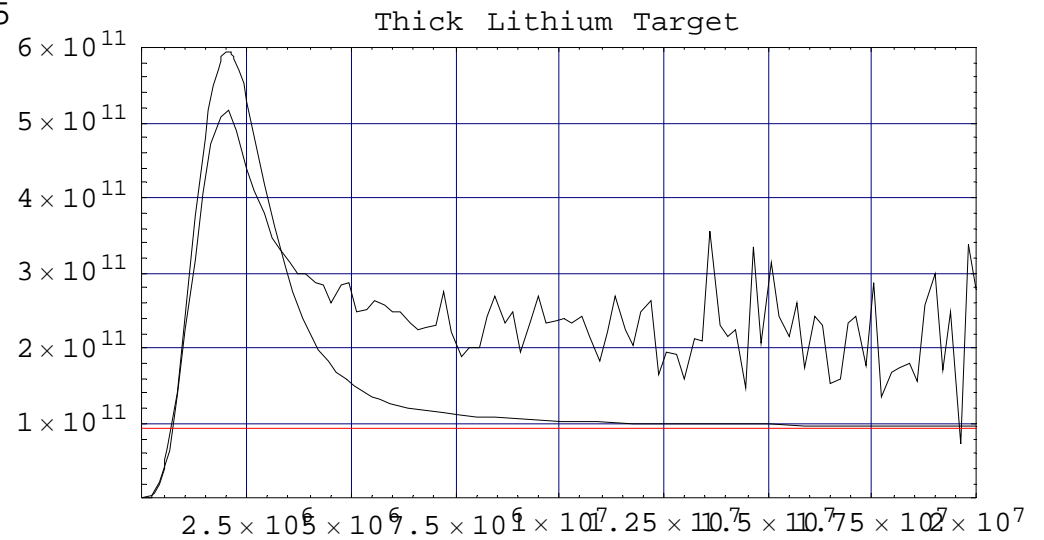
Tom Roberts has made some MC calculations for the MUSCAT targets using GEANT 4. The results are shown below:



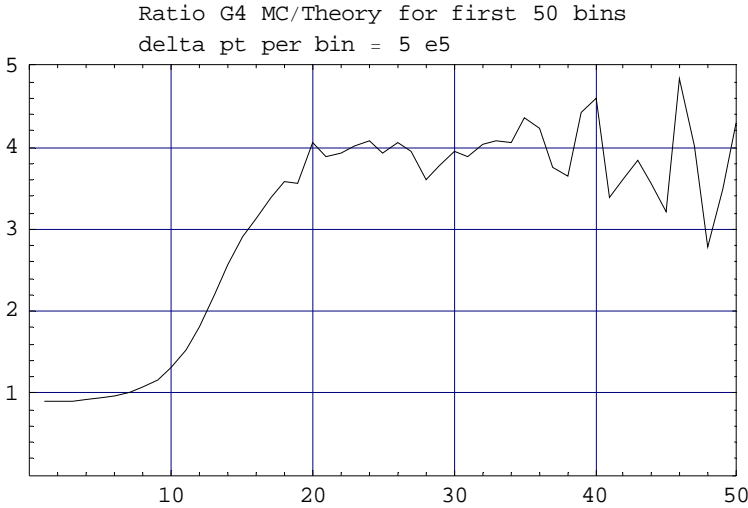
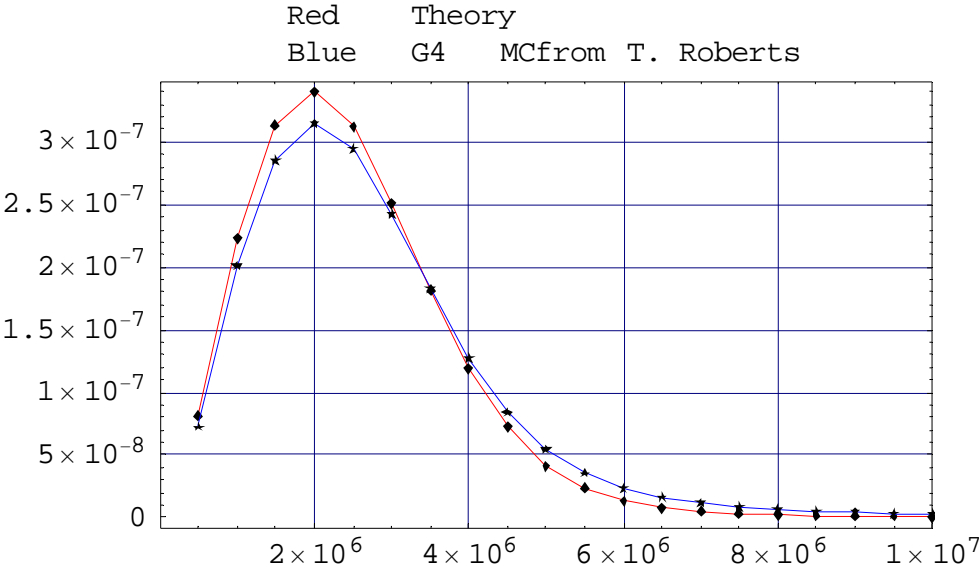
The plots are for projected p_x
The solid line is the theory as outlined
above. The red line is Rutherford



It is clear that there is trouble with GEANT. Part is explained by the use of Z ($Z+1$) instead of Z^2a factor of 2 for hydrogen. MUSCAT also observed this discrepancy.

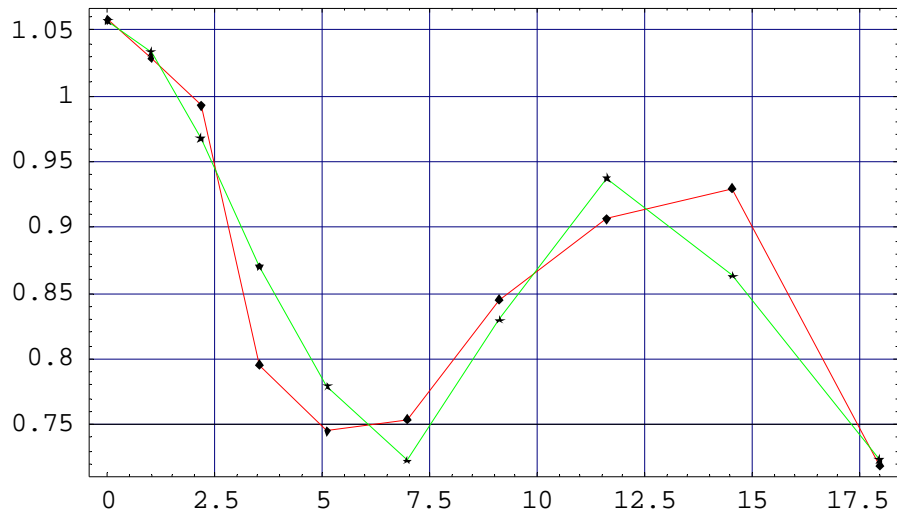


Comparison of G4 at small angles for MUSCAT 159 mm H target

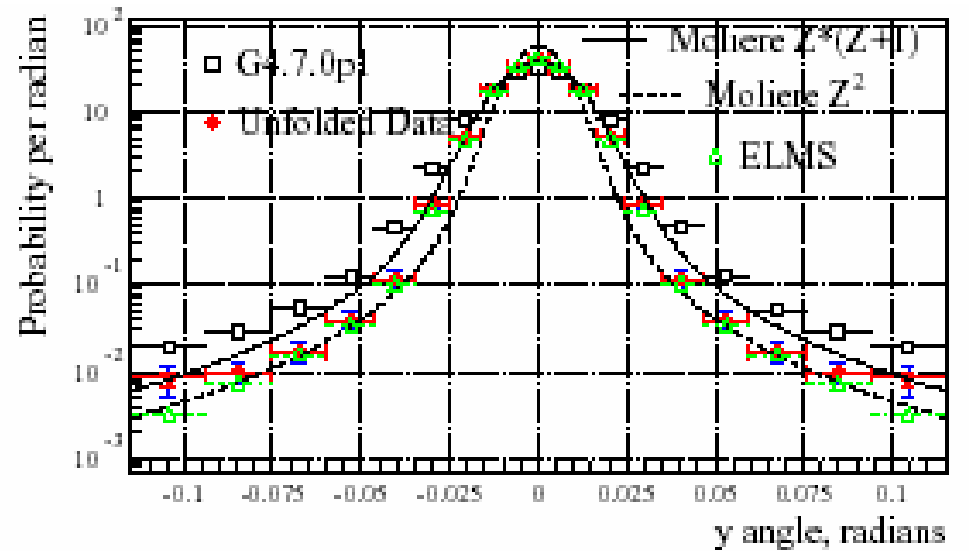
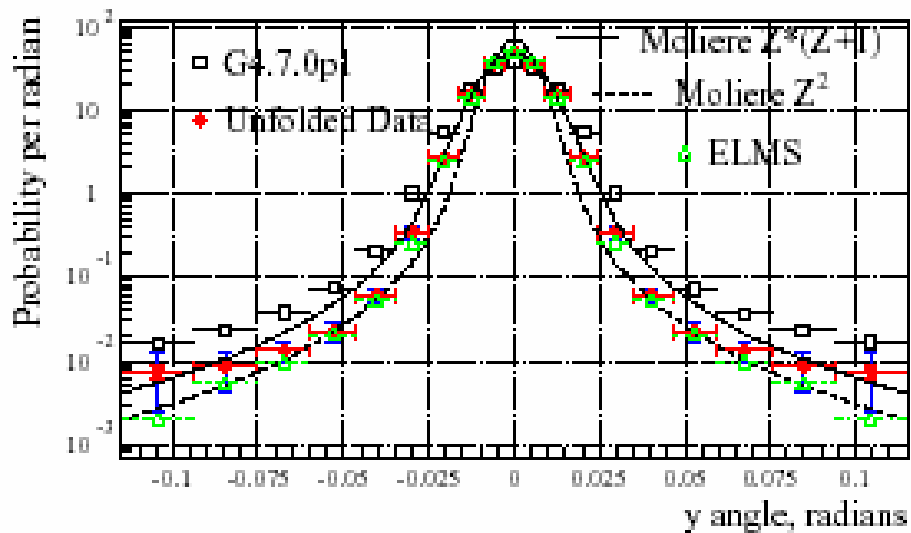


Wade Allison at Oxford has generated a new MC that not only corrects for the use of the Thomas Fermi model of the atom, but uses the correct basic scattering theory. This allows a calculation of the correlation between the energy loss and the scattering angle distribution. This correlation was observed in Neutrino Factory Study 1. It is not large but because the particle is moving in a magnet field it should be inserted correctly into the cooling MC. ELMS can do this, but I find an effect that doesn't seem to be correct. I plot below the ratio of the ELMS MC as given in the MUSCAT paper for the hydrogen targets to the theory presented here. The ratio should be 1 and in particular it must approach 1 for large scattering.

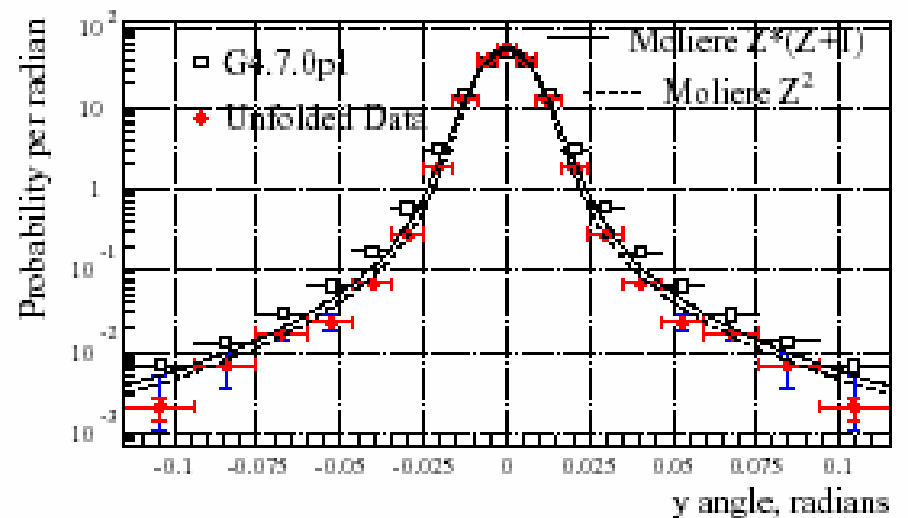
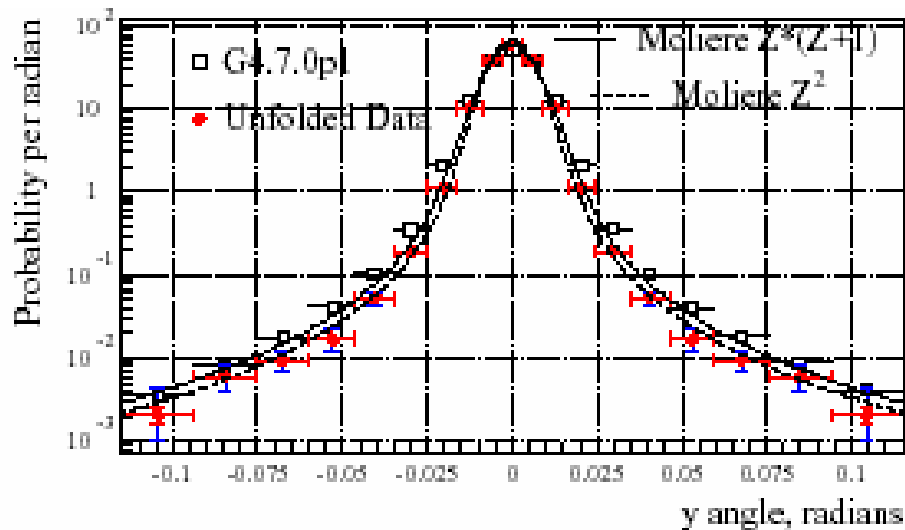
ELMS over theory for:
 Red 0.82 grms/cm² H
 Green 1.2 grms/cm² H



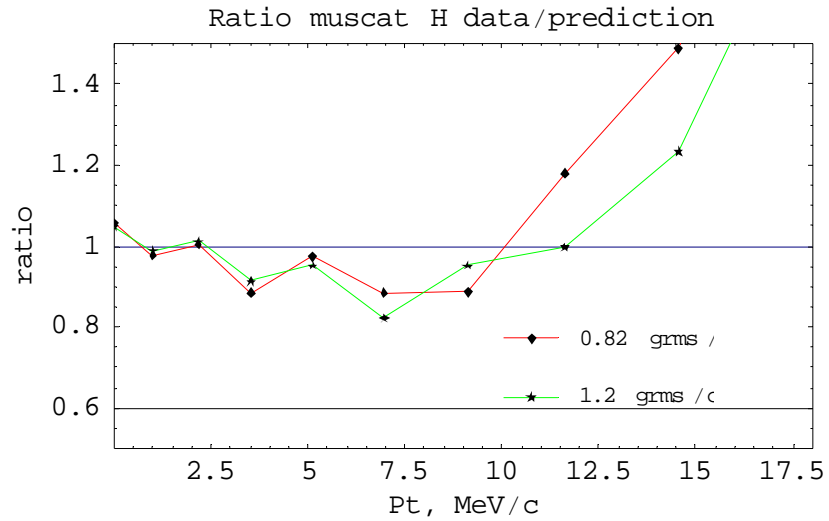
The horizontal scale is MeV/C and as shown above the scattering should just be Rutherford scattering off the proton for momentum transfers greater than 7 MeV/c.



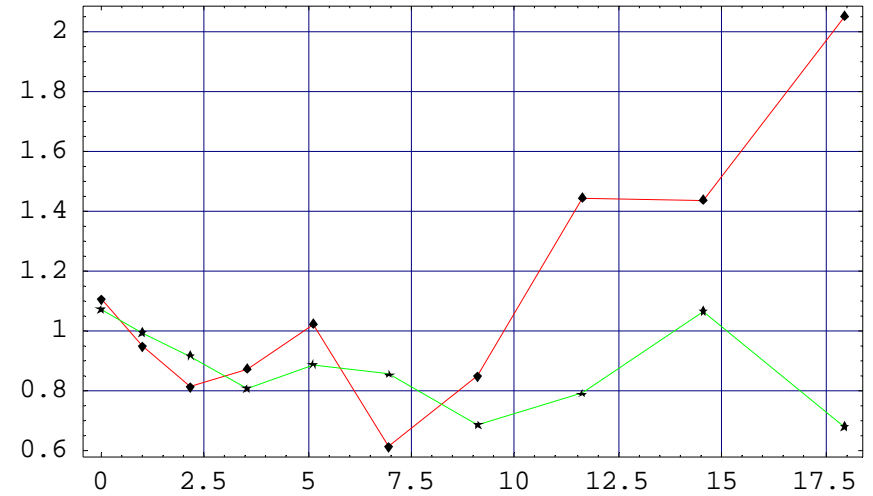
MUSCAT DATA



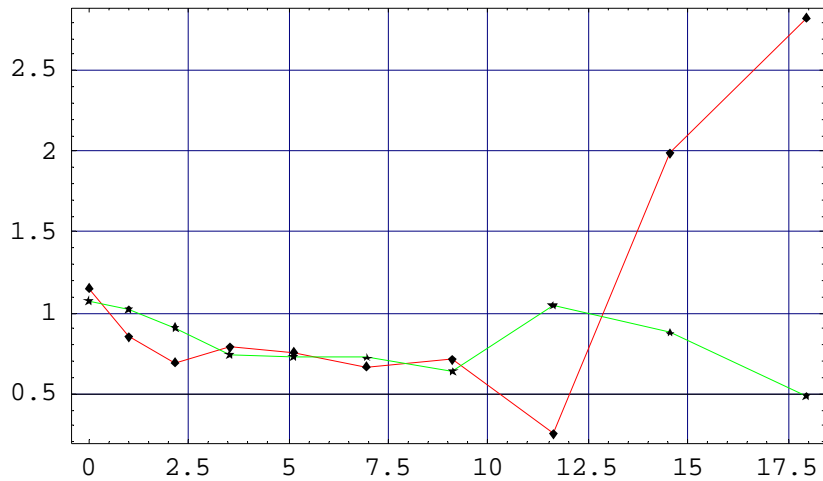
Experimental results from MUSCAT:



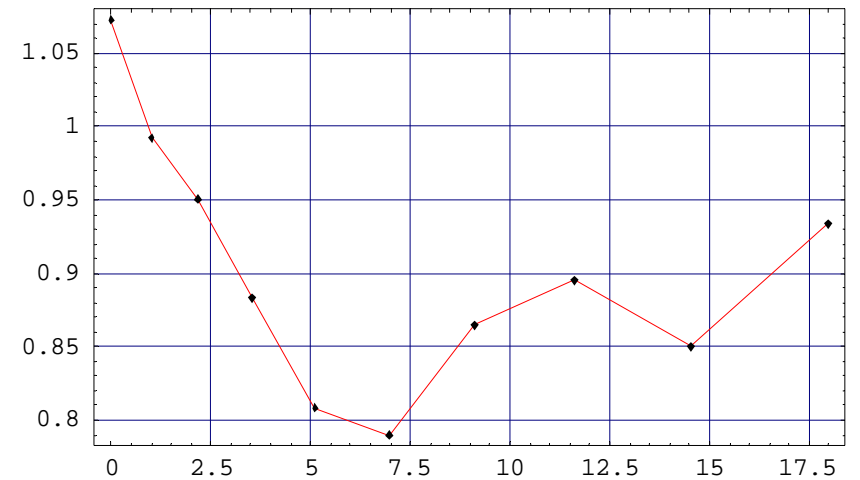
Ratio Exp/prediction for
 Red 0.34 grms/cm² Li
 Blue 0.67 grms/cm² Li
 Horizontal scale is in MeV/c

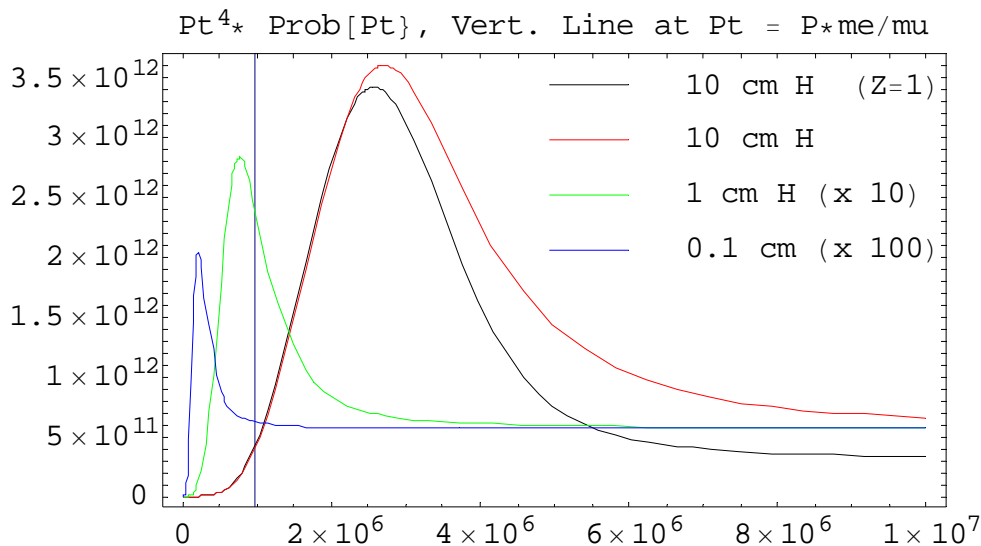


Ratio Exp/prediction for
 Red 0.18 grms/cm² Be
 Blue 0.69 grms/cm² Be
 Horizontal scale is in MeV/c



Ratio Exp/prediction for
 Red 0.42 grms/cm² C





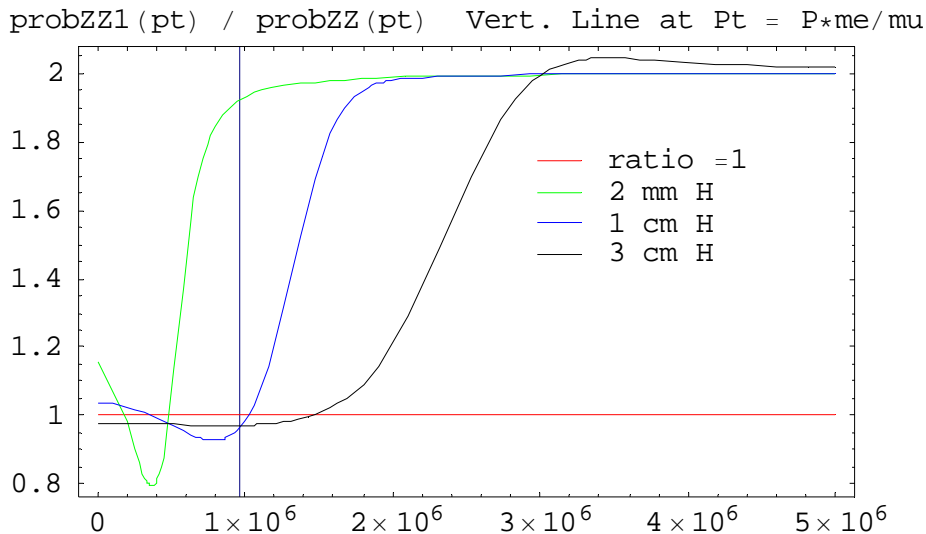
How is the transition made near:

$$\theta_{rms} \approx \theta_0 = \frac{m_e}{m_\mu} p \approx 1 \text{ MeV/c}$$

**For $\theta_{rms} \ll \theta_0$ use $Z(Z+1)$
but cut 1/B function by 0.5
at $\theta = \theta_0$.**

For $\theta_{rms} \gg \theta_0$ use Z^2 .

**The in between region
needs Monte Carlo study.**



CONCLUSIONS

1. We should get an accurate MC into the simulation programs.
2. G4 is off by a big factor for the larger angles which can influence particle loss and is slightly pessimistic at small angles.
3. The correlation between energy loss and large angle scattering makes small differences and was observed in Study I. ELMS or the work of Striganov could fix this.
4. The errors are small, but we need to clean things up.