MICE note nnnnn???
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Initial study of MICE Magnet Alignment Requirements

1 Introduction

Misalignments or displacements of the magnetic axes of the MICE focussing and coupling coils will result in the muons receiving $p_\perp$ kicks. Many small random $p_\perp$ kicks may lead to an increase in the emittance of the beam, though magnetic forces should not increase the emittance *per se*. Large $p_\perp$ kicks will steer the beam into walls and lead to losses. This note is a first attempt to quantify these effects and specify the constructional tolerances required for the MICE focus and coupling coils.

Order of magnitude arguments are presented in Section 2. The results of FS-Study-2 are presented in Section 3. Magnetic field errors are discussed in Section 4.3, the results of ICOOL simulations are discussed in Section 4 and preliminary alignment requirements are summarised.

2 Orders of magnitude

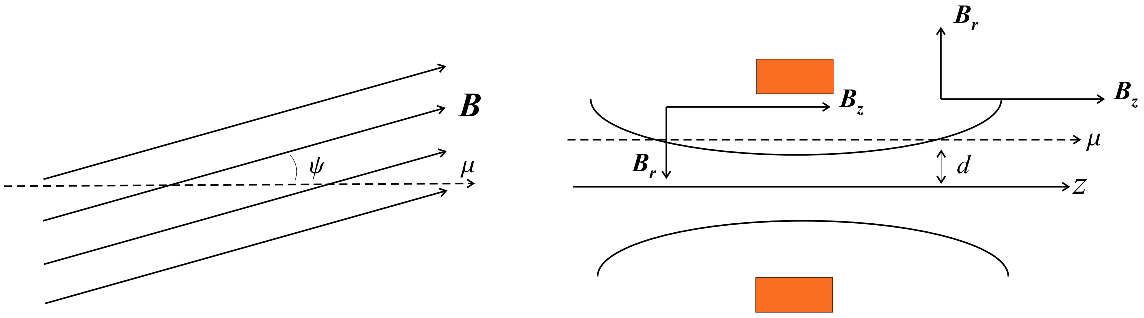

Figure 1: Left: A nominally on-axis muon (the reference particle) moving in the field of a tilted coil. Right: A nominally on-axis muon moving in the field of a displaced coil.

2.1 $p_\perp$ kicks

Consider a coil whose axis is either tilted by an angle ψ with respect to the axis of the system or displaced by a distance d, as sketched in Figure 1. A muon (the reference particle) moving in this field on the nominal (z) axis will receive a $p_\perp$ kick from the net transverse component, B_r, of the magnetic field. It is easy to convince one's self that $\int B_r dz = 0$ if there is a simple displacement (*i.e.* $\psi = 0, d \neq 0$). There is, however, a net $\int B_r dz$ if $\psi \neq 0$. To first order this is independent of d and the kick is $p_\perp = \int eB_r dz = 300(\mathrm{MeV}/c) \int B_r dz (\mathrm{T\,m})$. It can be shown relatively easily that

$$\begin{aligned} p_\perp &= \frac{300(\mathrm{MeV}/c)\psi}{2} \int_{-\infty}^{\infty} B_z dz \\ &\sim 150(\mathrm{MeV}/c)\mu_0 N I \psi \end{aligned}$$

where NI is the magnetising current. A pedestrian derivation of this result is given in the Appendix.

The MICE coils are relatively thin and can be regarded as 'thin' coils. In this approximation $B_z \propto (a^2 + z^2)^{\frac{3}{2}}$ where a is the radius of the coil, so the $p_\perp$ kick is localised to a region of length $\sim 2a$, *i.e.* ~50 cm for a focus coil, more for a coupling coil. The $p_\perp$ kicks can therefore be treated as approximately localised.

2.2 Beam losses

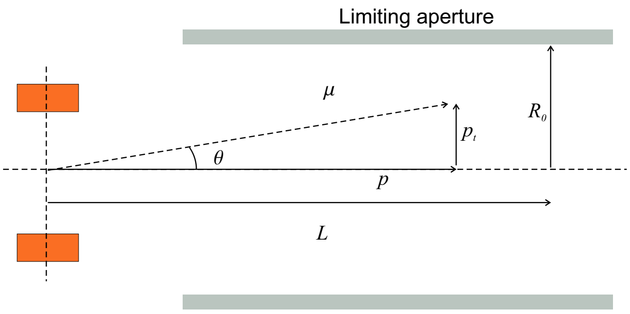

Figure 2: Displacement of reference particle due to small $p_\perp$ kick assumed to be located at centre of coil.

Consider the effect of the kick due to one tilted focus coil. As sketched in Figure 2, the reference particle will acquire a $p_\perp$ and be deflected by $\theta = p_\perp/p$ where p is the total momentum. This will steer the beam towards the edge of the aperture and losses will occur if the tails of the beam are scraped. One can estimate at what point this becomes important by considering the transverse size of the beam at the high-β_t waist at the centre of the RF cavities. The displacement of the reference particle here will be $\Delta x \sim L\theta = 150\mu_0 N I \psi L/p$ (p is in MeV/c)[1] where $L \sim 135$ cm.

At the high-β_t waist the rms size of the beam is $\sigma_y = \sigma_x = \sqrt{\beta_t \epsilon_{t,n} m_\mu / p} = 5.9$ cm with $\beta_t = 110$ cm, $\epsilon_{t,n} = 6000$ mm-mrad and $p = 200$ MeV/c. The confining radius, R_0, of the MICE channel (the cavity window apertures) is 21 cm at this point, *i.e.* $\sim 3.7\sigma_x$. The 15 cm radius of the absorbers also corresponds to $\sim 4\sigma_x$ at the low-β_t

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[1] $\mu_0 = 4\pi\, 10^{-7} \mathrm{H\,m^{-1}}$

waist of the $\beta_t = 42\,\text{cm}$ solution. This ratio of channel width to rms beam size will lead to losses of the unperturbed beam of $\int_{R_0/\sigma_x}^{\infty} \exp\left(-r^2/2\sigma_x^2\right) d(r/\sigma_x)^2 \sim 1/1000$ for the values given.

With a nominal ratio of aperture to beam rms of $(R_0/\sigma_x) \sim 4$, beam losses will start to become noticeable when $\Delta x \approx \sigma_x$, *i.e.* when the beam is clipped at 3σ on one side. Equating Δx due to the $p_\perp$ kick to σ_x, this will happen when

$$\psi = \frac{\sigma_x}{\mu_0 NIL} \frac{p}{150(\text{MeV}/c)}\,.$$

With $NI = 1.9\,10^6$ ampère turns for a focus coil and the other numbers as given above, this gives $\psi \approx 24\,\text{milliradians} = 1.4^\circ$. With $NI = 2.7\,10^6$ ampère turns for a coupling coil, the equivalent tilt is $\psi = 17\,\text{milliradians} = 1.0^\circ$; the beam would be clipped by the absorber body. The approximation of a localised kick may not be quite so good in this case. Since NI will scale with p these results should be independent of p for a fixed β_t solution.

The precise amount of loss as a function of ψ (or Δx) is difficult to estimate analytically since an asymmetric integration over a 2D gaussian distribution is required. Although this estimate is best left to MC simulations, it is worth noting that reducing the channel aperture symmetrically (*i.e.* radially) from 4σ to 3σ will introduce losses of $\sim 1\%$, so for this specific example, additional beam losses of $< 1\%$ are expected if one focus coil is tilted by $< 1.4^\circ$.

There will be greater sensitivity to tilts of the coupling coils because they are located in high-β_t regions: a deflection of θ at a coupling coil becomes $\sim \theta\sqrt{\beta_c/\beta_f}$ at the absorbers, where β_c and β_f refer to β_t at the coupling and focus coils respectively. For the 110,42 cm solution this amounts to an amplification factor of ~ 1.6.

Other cases which need to be considered are the tilting of the axis of a pair of focus coils in 'flip' and 'non-flip' (solenoid) modes. The flip mode should be forgiving because the $p_\perp$ kicks of the two coils will cancel; the non-flip mode will be unforgiving since the $p_\perp$ kicks will add. Roughly, in non-flip mode, significant beam losses would be expected for a tilt of the combined focus-pair axis of $\geq 0.7^\circ$ (12 mrad).

2.3 'Memory' of perturbations and averaging length

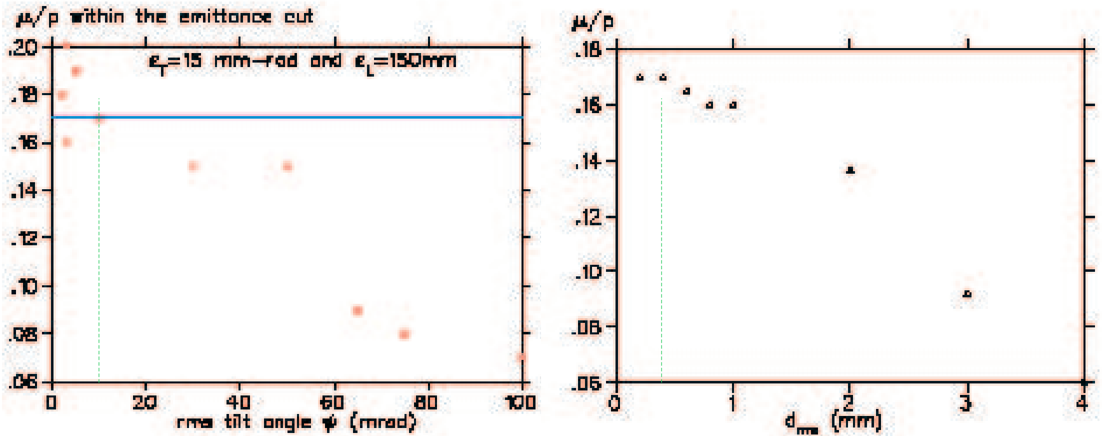
The $p_\perp$ kick received by a a muon will be reduced by its passage through the absorbers in the channel. Each 35 cm LH_2 absorber reduces the $p_\perp$ of a $200\,\text{MeV}/c$ reference muon by 6% so the $p_\perp$ acquired in one kick will be reduced by $1/e$ after $1/0.06 = 16.5$ absorbers. In other words a kick will be remembered on average for 16.5 half-lattice sections of focus coil, coupling coil and focus coil. We therefore take 32 focus coils and 16 coupling coils as the numbers over which to stochastically average the kicks for a long channel. MICE, of course, has fewer coils.

2.4 Conclusions of orders of magnitude estimates

The results of this semi-quantitative analysis are:

- Magnet displacements are expected to be relatively unimportant.
- Magnet tilts are important. A single tilted focus coil is expected to cause beam losses at the level of $\approx 1\%$ for a tilts of $> 1.4°$. Similar losses are expected if a focus coil is tilted by $> 1.0°$.
- Energy loss will damp out the effect of small magnetic field perturbations after typically 16 absorbers.
- The sensitivity of the performance of the channel to tilts is is expected to be greater for the coupling coils due to the higher magnetising currents and their location in regions of high β_t.

3 Feasibility Study 2 results



RMS Tilt Angle (milliradians)

Displacement (mm)

Figure 3: Study 2 ICOOL results

Some work on magnet alignment tolerances was reported in FS2[1]. There have also been MUCOOL studies[2] of a 15 T channel. The FS2 studies were made using both ICOOL and GEANT4. Figure 3 shows the ICOOL results in terms of μ/p within an acceptance cut ($\epsilon_{n,\perp} \leq 15$ mm-rad) at the end of the entire Study 2 cooling channel where the transverse emittance is 2.7 mm-rad. Within the statistical errors these results suggest that the rms tilt angle should be kept below approximately 10 mrad, but note that in the ICOOL studies magnets were displaced only every 5 m. The same results suggest that the rms displacements of the magnet centres must be kept below 0.3 – 0.5 mm. It is suggested that magnets can be built to these tolerances or better.

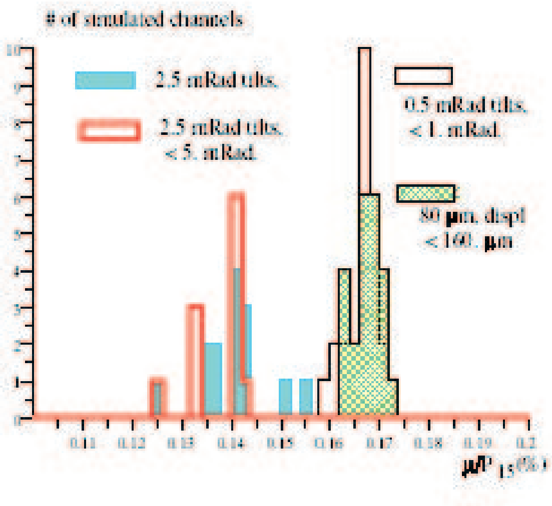


Figure 4: Study-2 GEANT-4 results.

The Study-2 results on tilts do not seem to be inconsistent with the arguments presented above, although the sensitivity to displacements is rather hard to understand.

The results from the GEANT4 studies quoted in Chapter 5 of FS2 seem to place more stringent requirements on the alignment tolerances; an rms polar tilt of 2.5 mrad seems to be unacceptable. The MUCOOL studies suggest even tighter tolerances, probably because the magnetic fields were several times higher.

4 Simulations

In addition to the order of magnitude estimates described in Section 2, we have performed a series of MC runs to determine the acceptable tolerances in the alignment of the MICE magnet coils. For this purpose we used version 2.57 of the the ICOOL package [3] with an idealised geometry and ideal gaussian beam.

4.1 Monte Carlo Setup

In our MC, we used the geometry from the MICE Proposal to RAL dated 10 January 2003 [4]. In general, in our representation the elements of MICE had a radius of 21.3 cm, reduced to 15 cm in the LH_2 sections. We inserted all 18 MICE magnet coils into our ICOOL-based program through an external CELL field. Magnetic fields were defined as a BLOCK structure. The dimensions and locations of each coil were taken from the MICE Proposal, Table 4.1. The current densities were the ones from Table 4.2 for case 1a (first column).

The only major difference between the actual structure of MICE and the geometry used in our simulations was the design of the aluminum absober windows. For simplicity, in our MC we employed cylindrical LH_2 absorbers of length 35.3 cm. Consequently the aluminium windows were flat disks with a uniform thickness of 180 μm for the absorber windows and 140 μm for the vacuum windows.

4.2 RF Cavities

No document providing a complete description of the MICE RFs was available, but as we shall see, a perfect representation is not even necessary for our study. Thus, we chose to experiment with four different RF configurations. We tried: no electric field at all; a uniform and time-independent electric field (ICOOL ACCEL model 1, mode 0); a uniform sinusoidal field with frequency 201.25 MHz (ACCEL model 1, mode 1), and a pillbox RF field in the TM_{010} mode with frequency 201.25 MHz (ACCEL model 2, mode 0). In the two latter cases, we tuned the RFs for both on-crest and off-crest (phase angle 45°) operation and varied the peak electric field accordingly.

We carried out some preliminary simulation runs to determine the RF model that was most suitable for our purposes. When using no electric field at all, we found that normalised transverse emittance $\epsilon_{n,t}$ tended to diverge after the beam emerged from the last LH_2 absorber. This was probably because the shape of the beam in phase space was becoming non-gaussian towards the end of the MICE channel and $\epsilon_{n,t}$ was computed by `ecalc9f` [5] from rms values. Our simulations also showed a negligible difference between uniform and pillbox RFs. Additionally, there was no major difference between time-independent RFs and properly tuned time-dependent off-

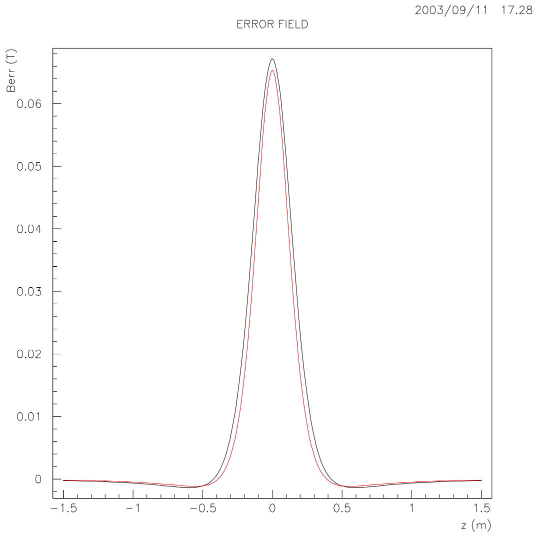


Figure 5: Error in B_y as a function of z for a focus coil rotated by $\psi = 1°$ around the x axis. The black curve is the error field from OPERA; the red line is the analytical estimate assuming a thin coil with radius 27.5 cm and current $1.9\,10^6$ A.

crest RFs. On the other hand, the tuning of on-crest time-dependent RFs was more problematic and so far we have been unable to achieve results that are completely satisfactory.

For the time being, we selected a uniform and time-independent electric field (ACCEL model 0, mode 0). The length of each RF module in our MC is 43 cm and the gradient on-axis at the centre of the gap is 8.0 MV/m. The total acceleration provided by one RF section is thus 13.76 MeV/c, *i.e.* about the average momentum lost in each LH_2 absorber by a 200 MeV/c muon.

4.3 Magnet misalignment and field errors

So far we have studied the misalignment of individual coils only. In each MC run, a single coil (either focussing or coupling) was either tilted with respect to the the longitudinal (z) axis or displaced in the transverse direction. The study of the misalignment of a pair of focussing coils and the random misalignment of all coils in a single MC run will be conducted in the near future.

The magnetic field grid generator OPERA[6] was used to generate magnetic field grids due misaligned coils. Specifically grids were generated which contained the difference between the field of a displaced or tilted coil and the field from the same coil correctly aligned. These differences were then read into ICOOL as BACKGROUND fields and added to the CELL field that represented the correctly aligned coil structure.

The grids from OPERA were 3 m long with a 5 cm spacing, *i.e.* with 61 points along the longitudinal axis. Along both transverse directions, we used a 40 cm width, ranging from -20 cm to 20 cm, with a step-size of 1.38 cm (*i.e.* with 30 points along each direction). We generated grids for $\psi = 1^\circ$ tilts and displacements of $d = 1$ cm and 1 mm. For intermediate values we used a linear interpolation of the data from these OPERA grids. Figure 5 shows the transverse (vertical) field error (ΔB_y) from an OPERA field map as a function of z for a focus coil rotated by $\psi = 1^\circ$ around the x (horizontal) transverse axis. Also shown is the result of the first order approximation discussed in Section 2.1. The two curves are in excellent agreement. The narrowness of the curve supports the approximation that the $p_\perp$ kicks due to field errors can be treated as localised.

4.4 Input beam

An idealised gaussian input beam was generated at the centre of the upstream spectrometer solenoid (Spectrometer Solenoid 1 MICE Proposal, Figure 3.1). We generated muons with longitudinal momentum of $p_z = 200$ MeV/c, with no dispersion in z, p_z or t. So far we have chosen not to investigate the longitudinal emittance, $\epsilon_{||}$, and hence the starting beam had no spread in longitudinal phase space.

In the transverse phase space, we assumed a gaussian beam with an initial normalised transverse emittance of $\epsilon_{n,t} = 6000$ mm-mrad and an initial beta transverse of $\beta_{n,t} =$ 33 cm. The initial beam parameters were $\sigma_x = 3.3$ cm and $\sigma_{p_x} = 20$ MeV/c and similarly for σ_y, σ_{p_y}. Since the beam was generated in the middle of a 4 T solenoidal field, the appropriate solenoidal kick was added to the transverse momenta. We used no other correlations in the generation of the beam. For simplicity, the muon decay mode was disabled in ICOOL, although ~1% of the muons are expected to decay in the MICE channel.

4.5 Criteria

We used two quantities to determine the acceptable tolerances in magnet alignment. Specifically, we considered beam transmission and transverse emittance variations from the middle of the upstream tracker to the middle of the downstream tracker (Spectrometer Solenoids 1 and 2, MICE Proposal, Figure 3.1). The criteria we chose to use were:

- The number of particles lost due to coil misalignment should be small compared to the number inherently lost during propagation through a channel with correctly aligned magnets. Some particles, about 1% of the total for our input beam, are lost in the correctly aligned case. (The reason for these losses has not been investigated yet.) Another 1% are expected to be lost by decay. Therefore, we require that the total number of particles lost with misaligned coils be within the statistical error of the number lost in the correctly aligned case.

- Emittance decrease (*i.e.* cooling) should deteriorate by at most about 1%. We define emittance decrease $\Delta\epsilon/\epsilon$ as $(\epsilon_{n,t,1} - \epsilon_{n,t,2})/\epsilon_{n,t,1}$, where the indices 1, 2 refer to the upstream and downstream trackers, respectively. Assuming *e.g.* that $\Delta\epsilon/\epsilon = 15\%$ in the ideal case, we require that $\Delta\epsilon/\epsilon$ become no smaller than 14% due to misalignment.

The MICE Proposal in Section 3.4.2 calls for a deterioration of $\Delta\epsilon/\epsilon$ of at most one part in 1,000 absolute. As we shall see in the next Section, at this stage we are not able to validate such a stringent requirement due to limits in our MC event statistics (and it will be difficult ever to do so).

4.6 Simulation results

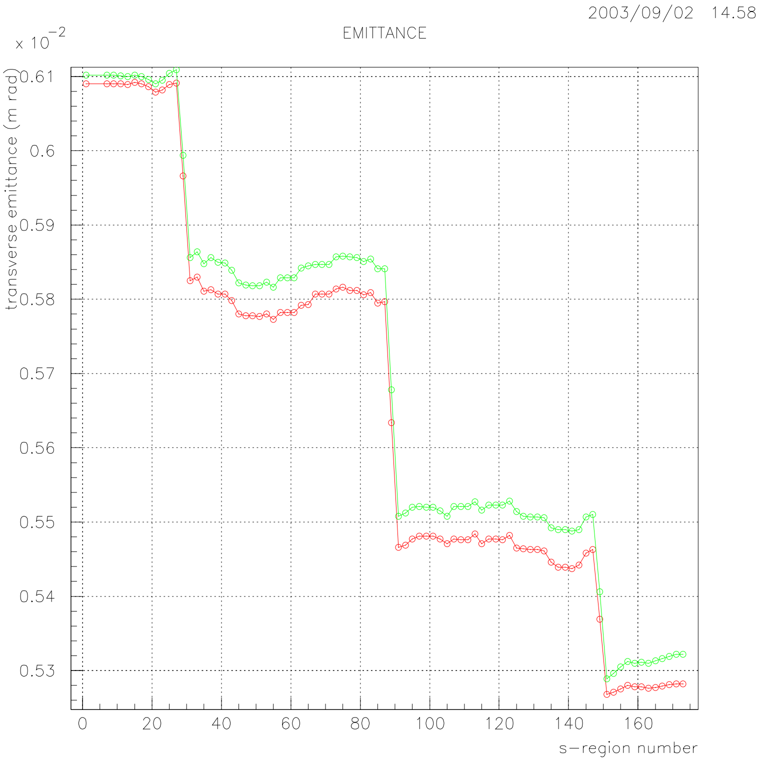


Figure 6: $\epsilon_{n,t}$ (m-rad) versus position along the longitudinal (z) axis. The red line is for correctly aligned magnets; the green line for a $\psi = 0.8°$ tilt of focus coil number 4.

After a series of preliminary test runs with limited statistics, we decided to study the displacement of focussing coil number 4, *i.e.* the downstream coil around the central LH_2 absorber. This appeared as a reasonable choice for the time being, in order to produce some average results. We intend eventually to check all coils. It is worth mentioning that the preliminary runs show that the misalignment of a downstream focus coil of a pair has a larger effect on beam deterioration than the corresponding upstream focus coil.

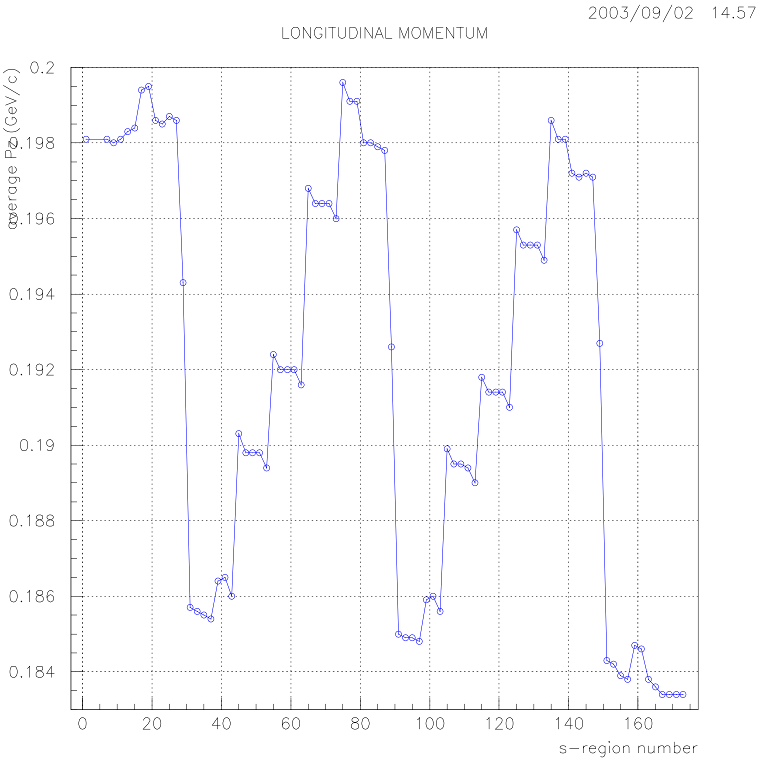


Figure 7: p_z (GeV/c) along the longitudinal z axis. The three LH$_2$ absorbers and two RF regions can be clearly identified.

10,000 events were simulated in each MC run. This number was a compromise between statistical error and CPU time (it took about 24 hours to run and process each 10,000 event sample). We studied tilts ranging from $\psi = 0.1°$ to $10°$ and lateral displacements from $d < 1$ mm to > 5 cm.

Figure 6 shows the evolution of the transverse normalised emittance $\epsilon_{n,t}$ along the MICE channel, from the upstream tracker to the downstream tracker. The longitudinal axis is divided into s-regions, which represent the subdivisions along the z axis used in our MC code and loosely follow the metrics of this axis. The red curve is for the correctly aligned case and the green curve is for a $\psi = 0.8°$ tilt of focus coil number 4. We can see cooling taking place in the three LH$_2$ absorbers and a roughly constant $\epsilon_{n,t}$ elsewhere. In this figure, from the values of the first and last data point on each curve, we achieve a cooling of $\Delta\epsilon/\epsilon = 13.3\%$ in the correctly aligned case and $\Delta\epsilon/\epsilon = 12.8\%$ for a $\psi = 0.8°$ tilt.

Figure 7 shows the variation of p_z along the MICE channel. Again, three large decreases in p_z correspond to the passage of the beam through the LH$_2$ absorbers and the two areas where p_z increases correspond to the two RF sections (with 4 RF cavities in each section).

Figures 8 and 9 show the number of events lost (out of an initial 10,000) and $\Delta\epsilon/\epsilon$ at the end of the MICE channel respectively as a function of tilt angle. The 10% statistical fluctuations in the loss for small tilts are simply fluctations on the ~100

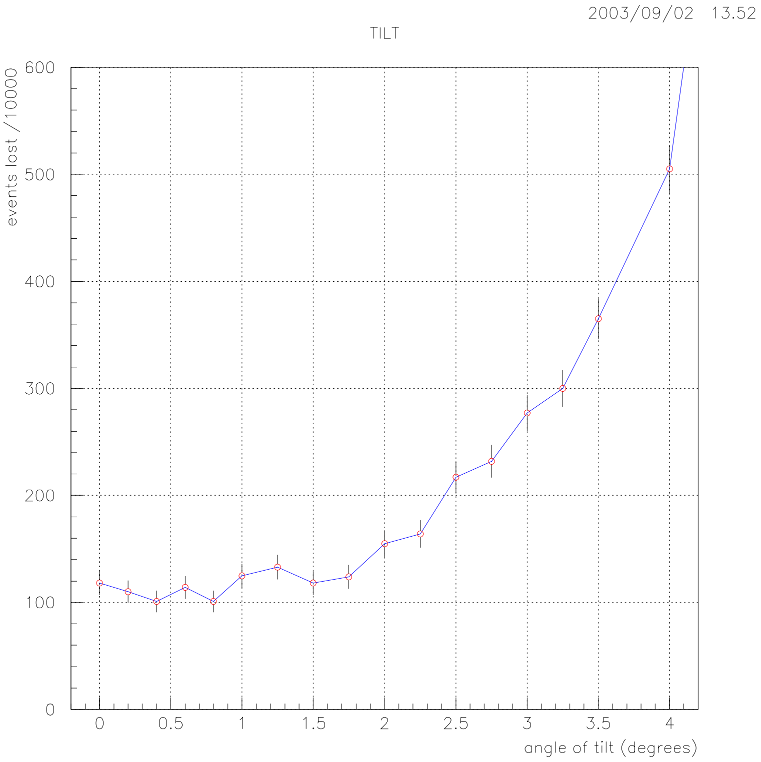


Figure 8: Number of events lost (out of initial 10,000) at the end of MICE vs. angle of tilt (°) of focus coil number 4.

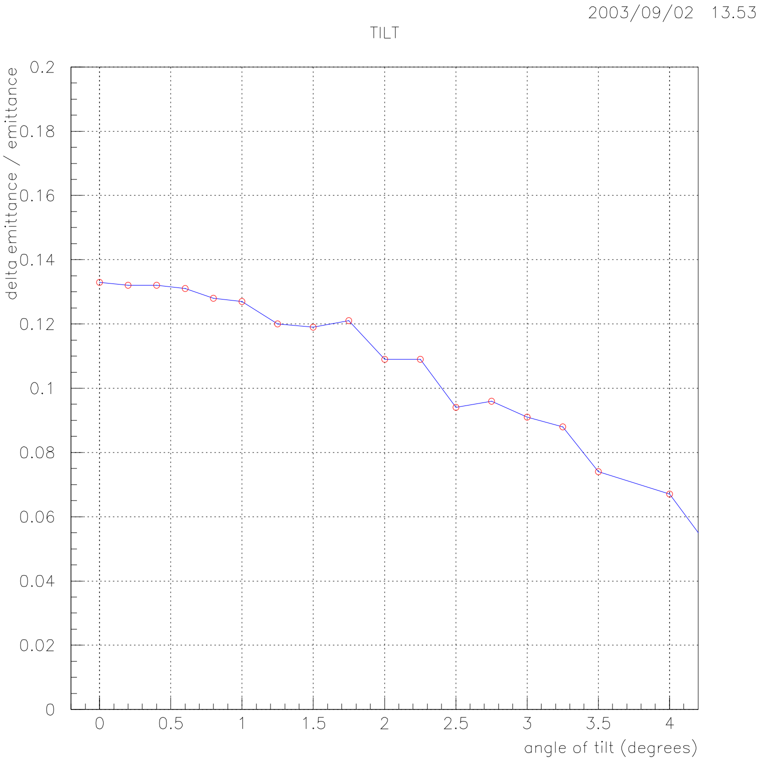


Figure 9: $\Delta\epsilon/\epsilon$ versus angle of tilt (°) of focus coil number 4.

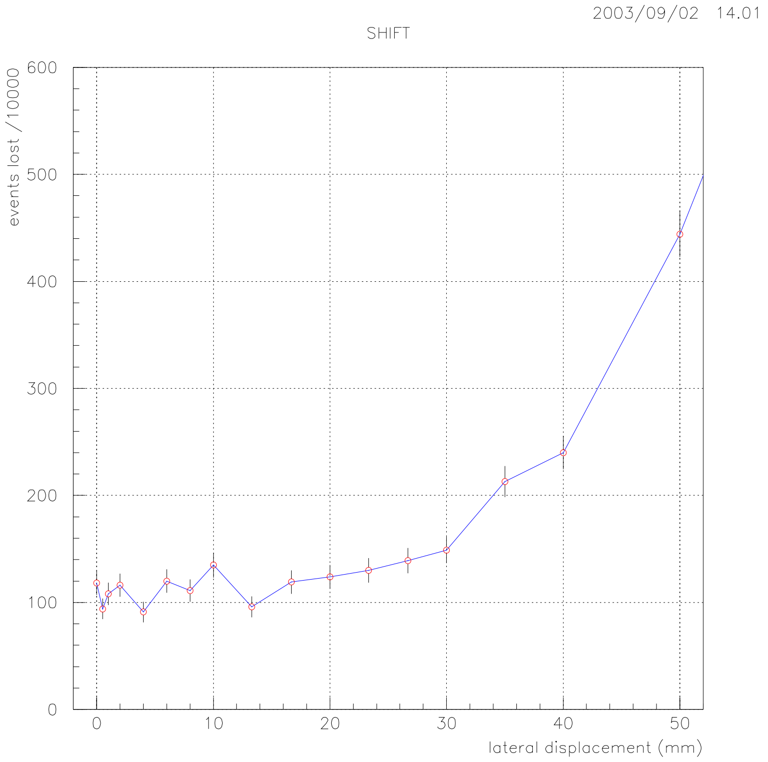


Figure 10: Number of events lost (out of initial 10,000) at the end of MICE vs. transverse displacement (mm) of focus coil number 4.

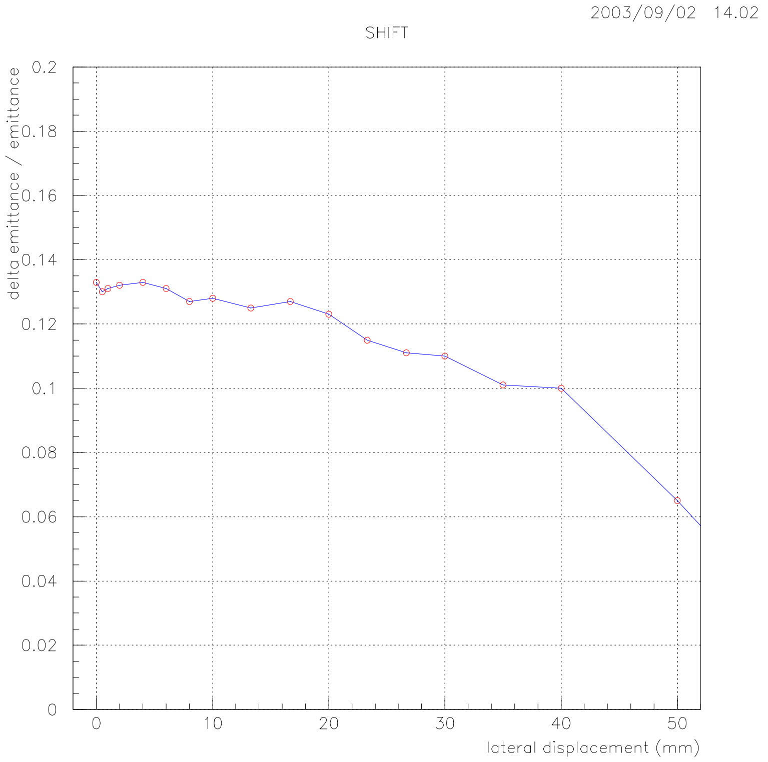


Figure 11: $\Delta\epsilon/\epsilon$ vs. transverse displacement (mm) of focus coil number 4.

particles lost from the initial 10,000. Both estimators (loss and emittance) show significant deterioration in performance for tilts of $> 1.5^\circ$. Some of the deterioration of apparent cooling performance is simply that the scraping of the beam results in some effective collimation; the input emittance of the surviving beam particles is less than the emittance of the initial beam. The deterioration in transmission observed for tilts of 1.5° and above agrees well with the estimates given in Section 2.

From the criteria described in Section 4.5 above, we choose $\psi = 1.5^\circ$ as the maximum acceptable tilt for *one* focus coil.

Figures 10 and 11 show event loss and $\Delta\epsilon/\epsilon$ deterioration as a function of transverse displacement. In this case, a displacement of $d \leq 2$ cm seems acceptable. As discussed in Section 2, transverse displacement should have no effect in first approximation. These MC results confirm the analytical estimate.

5 Preliminary Conclusions

Results from the previous sections apply to the misalignment of one individual focus coil only. The part of the MICE experiment that we are interested in contains a total of eight coils (6 focussing and 2 coupling). Therefore, given that there are fewer coils than the averaging length of 16 half-sections, the tolerances on the misalignment for all the focus coils can be estimated by dividing the values given above by $\sqrt{6}$. This of course is only an approximation, based on the assumption that the effects of each misaligned coil are random and independent. Better estimates will become available in the future by performing MC runs where multiple magnet coils are misaligned, but this is a time-consuming process.

Our preliminary conclusions are that any tilt of the focus coils alone must be kept below $1.5^\circ/\sqrt{6} \approx 0.6^\circ \approx 10$ mrad. The transverse displacement of the axis of the coil must be below $d < 20$ mm / $\sqrt{6} \approx 8$ mm.

For the reasons given in Section 2.2, there is expected to be greater sensitivity to misalignment of the coupling coils by a factor of $\sim (NI_c/NI_f)\sqrt{\beta_c/\beta_f} = (2.7/1.9) \times \sqrt{110/42} = 2.3$. Treating all eight coils democratically, we arrive at $\psi_{\text{max}} = 1.5^\circ/\sqrt{2 \times 2.3^2 + 6} = 1.5^\circ/4.1 = 0.37^\circ = 6.4\,\text{mrad}$ for *any* of the coils. Similarly the maximum displacement will be $d_{\text{max}} = 20/4.1 = 4.9\,\text{mm}$. These do not seem to be demanding.

The figures given above refer to MICE. The MICE coils should, however, be designed 'for real' *i.e.* as if they are part of a real cooling channel or ring. One lattice section contains four focus and two coupling coils. The damping length for perturbations is estimated in Section 2.3 to be 16.5 half-lattice sections so the averaging factor (if it legitimate to treat the perturbations as stochastic) becomes $1/\sqrt{16.5(2.3^2 + 2)} = 0.091$. The maximum tolerable tilts and displacements then become $\psi_{\text{max}} = 0.13^\circ = 2.4\,\text{mrad}$ and $d_{\text{max}} = 1.8\,\text{mm}$. We believe the MICE coils should be constructed to be within these tolerances, which would not seem to present great constructional challenges.

Longitudinal displacement has not been simulated yet, but tolerances are expected to be less stringent than the ones on lateral displacement.

6 Acknowledgments

We would like to thank Holger Witte of Oxford Condensed Matter Physics for generating the OPERA magnetic field grids.

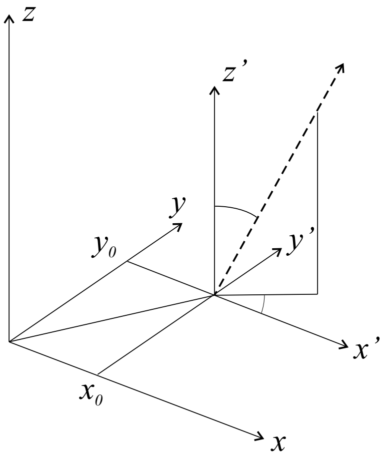


Figure 12: Coordinate system used for derivation of $p_\perp$ kick experienced by a muon. z, shown vertically, is the central beam axis.

Appendix

Transverse momentum kick from a displaced and tilted coil

Imagine that the centre of a coil is displaced in the transverse plane by $-x_0$ and $-y_0$ from the nominal position and the axis of the coil is tilted by a polar angle ψ and rotated by an angle ϕ around the z (beam) direction. The geometry is shown in Figure 12. In the frame of the coil, if ψ is small, a muon moving on the nominal z axis has a trajectory $x = x_0 + z\psi\cos\phi$ and $y = y_0 + z\psi\sin\phi$. The tilt and displacement of the coil will result in the muon receiving a $p_\perp$ kick.

Assuming that the deflection is small, and that the muon remains close to the axis, the $p_\perp$ kick will be

$$\Delta p_\perp \sim k \int \hat{\vec{v}} \wedge \vec{B} dz$$

where $k = 300$ MeV/c $\mathrm{T}^{-1}\mathrm{m}^{-1}$. To first approximation the radial component of the magnetic field seen by the muon is (from $\nabla \cdot \vec{B} = 0$)

$$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}$$

giving

$$B_x = \frac{x}{r} B_r = -\frac{x}{2}\frac{\partial B_z}{\partial z}$$

and

$$B_y = -\frac{y}{2}\frac{\partial B_z}{\partial z}.$$

The vector product within the integral is

$$\hat{\vec{v}} \wedge \vec{B} \;\sim\; \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \psi\cos\phi & \psi\sin\phi & 1 \\ -\frac{x}{2}\frac{\partial B_z}{\partial z} & -\frac{y}{2}\frac{\partial B_z}{\partial z} & B_z \end{vmatrix} .$$

The resultant momentum kick in the x direction is

$$\begin{aligned} \Delta p_x &= k\int_{-\infty}^{\infty}\left\{\psi\sin\phi B_z + \frac{1}{2}(y_0 + z\psi\sin\phi)\frac{\partial B_z}{\partial z}\right\}dz \\ &= k\psi\sin\phi\int_{-\infty}^{\infty}B_z dz + 0 + \frac{k}{2}\psi\sin\phi\int_{-\infty}^{\infty}z\frac{\partial B_z}{\partial z}dz \end{aligned}$$

where the integral of the second (y_0) term is zero because $\int\frac{\partial B_z}{\partial z}dz = 0$. The last term may be integrated by parts:

$$\begin{aligned} \int_{-\infty}^{\infty}z\frac{\partial B_z}{\partial z}dz &= [zB_z]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}B_z dz \\ &= -\int_{-\infty}^{\infty}B_z dz \end{aligned}$$

where it is assumed that B_z falls faster than z^{-1} so that the first term is zero, which is certainly true for a thin coil. On the axis of a circular (or cylindrical) coil $\int_{-\infty}^{\infty}B_z dz = \mu_0 NI$, where NI ampère-turns is the magnetising current in the coil, follows from Ampères law (and can be shown explicitly for a thin coil). Collecting things together, the net p_x imparted to the muon is

$$\begin{aligned} \Delta p_x &= \frac{k\psi\sin\phi}{2}\int_{-\infty}^{\infty}B_z dz \\ &\sim \frac{k\psi\sin\phi\mu_0 NI}{2} . \end{aligned}$$

Likewise the net p_y impulse is

$$\Delta p_y \sim -\frac{k\psi\cos\phi\mu_0 NI}{2} .$$

The net transverse momentum squared is

$$\begin{aligned} \Delta p_\perp^2 &= \Delta p_x^2 + \Delta p_y^2 \\ &= \left(\frac{k\psi\mu_0 NI}{2}\right)^2(\sin^2\phi + \cos^2\phi) \\ &= \left(\frac{k\psi\mu_0 NI}{2}\right)^2 . \end{aligned}$$

Within the approximations made here, $\Delta p_\perp^2$ depends only on ψ, the angle between the magnetic axis of the coil and the nominal beam direction; there is no dependence on the displacements x_0 and y_0 of the centre of the coil.

A muon on the nominal z axis passing through some (large) number, M, of tilted coils will acquire a mean $p_\perp^2$ of

$$\Delta p_\perp^2 \quad = \quad M \left(\frac{k\mu_0 NI}{2} \right)^2 \langle \psi^2 \rangle$$

where $\langle \psi^2 \rangle$ is the mean square tilt angle. The mean $p_\perp^2$ projected onto a transverse plane, $\Delta p_{x,y}^2$, will be

$$\Delta p_{x,y}^2 \quad = \quad \frac{M}{2} \left(\frac{k\mu_0 NI}{2} \right)^2 \langle \psi^2 \rangle \, .$$

References

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